F-theory on tetrahedron

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Abstract. *Complex tetrahedral surface $\mathcal{T}$ is a non planar projective surface that is generated by four intersecting complex projective planes CP^2. In this paper, we study the family $\{\mathcal{T}_m\}$ of blow ups of $\mathcal{T}$ and exhibit the link of these $\mathcal{T}_m$s with the set of del Pezzo surfaces dP_n obtained by blowing up n isolated points in the CP^2. The $\mathcal{T}_m$s are toric surfaces exhibiting a $U(1) \times U(1)$ symmetry that may be used to engineer gauge symmetry enhancements in the Beasley-Heckman-Vafa theory. The blown ups of the tetrahedron have toric graphs with faces, edges and vertices where may localize respectively fields in adjoint representations, chiral matter and Yukawa tri-fields couplings needed for the engineering of F- theory GUT models building.*
Key words: F-theory on Calabi-Yau 4-folds, del Pezzo surfaces, BHV model, Intersecting Branes, Toric singularites.

1. Introduction

With the advent of the Large Hadron Collider (LHC) at CERN, theoretical studies around the Minimal Supersymmetric Standard model (*MSSM*) and Grand Unified Theories (GUT) have known intense activities. Among these research activities, the studies of TeV- scale decoupled gravity scenarios aiming the embedding of *MSSM* and GUT models into superstrings and M- theory [1, 2, 3, 4]; see also [5, 6, 7, 8, 9, 10]. Recently Beasley-Heckman-Vafa made a proposal, to which we refer here below as the BHV model, for embedding *MSSM* and GUT into the 12D F-theory compactified on Calabi-Yau four- folds [11, 12, 13]. In this proposal, the visible supersymmetric gauge theory in 4D space time including chiral matter and Yukawa couplings is given by an effective field model following from the supersymmetric gauge theory on a seven brane wrapping 4-cycles in the F-theory compactification down to *4D* Minkowski space time. In the engineering of the supersymmetric GUT models in the framework of the BHV theory [12, 13], see also [14, 15], one has to specify, amongst others, the geometric nature of the complex base surface S of the elliptically K3 fibered Calabi-Yau four -folds X_4:

$$\begin{array}{ccc} Y & \rightarrow & X_4 \\ & & \downarrow \pi_s \\ & & S \end{array} \tag{1.1}$$

In this relation Y is a complex two dimension fiber where live ADE singularities giving rise to the rank r gauge symmetry G_r that we observe in 4D space time and S is a complex base surface whose cycle homology captures important data on matter fields representations and their tri- fields couplings. If the singular fiber Y of the local Calabi-Yau four-folds (CY4) is fixed by the targeted 4D space time invariance G_r, one may a priori imagine several kinds of compact complex surfaces S as its base manifold. The choice of S depends on the effective 4D space time physics; in particular the number of conserved supersymmetric charges and chiral matter fields as well as their couplings. Generally speaking, the simplest surfaces one may consider are likely those given by the so called Hizerbruch surfaces

$$F_n = CP^1 \times_n CP^1$$

generated by fibration of a complex projective line over a second projective line. Other examples of surfaces are given by the complex projective plane CP^2 and its del Pezzo dP_n cousins; or in general non planar complex surfaces $\mathcal{D}$ embedded in higher dimensional complex Kahler manifolds. Typical examples of adequate surfaces S that have been explicitly studied in the BHV model are given by the family of del Pezzo surfaces dP_n with $n = 0, 1, ..., 8$. These complex surfaces are obtained by preforming up to eight blow ups at isolated points of the projective plane $CP^2 = dP_0$ by complex projective lines [16, 17, 11, 18, 19]; see also section 2 for technical details.

Motivated by the study of the geometric engineering of the F-theory GUT models building à la BHV, we aim in this paper to contribute to this matter by constructing a family of backgrounds for F-theory compactification based on the tetrahedron geometry $\mathcal{T}$ and its blow ups. This study sets up the basis for developing a class of F-theory *GUT- like* models building and uses the power of toric geometry of complex surfaces to geometrically engineer chiral matter and the Yukawa couplings. Recall that the tetrahedron $\mathcal{T}$ viewed as a toric surfaces with the following toric fibration

$$\begin{array}{ccc} T^2 & \rightarrow & \mathcal{T} \\ & & \downarrow \pi_\Delta \\ & & \Delta_\mathcal{T} \end{array} \tag{1.2}$$

has toric singularities generated by shrinking cycles of T^2 on the edges of the tetrahedral base $\Delta_\mathcal{T}$ and at its vertices. In our approach, the shrinking cycles of the above toric fibration are interpreted in terms of gauge enhancement of bulk gauge symmetry $G_r \times U^2(1)$. In going from a generic face of the tetrahedron towards a vertex passing through a edge, the $G_r \times U^2(1)$ bulk gauge symmetry gets enhanced to $G_{r+1} \times U(1)$ on the edge and to G_{r+2} at the vertex as shown on the following table:

Tetrahedron $\mathcal{T}$	:	faces	edges	vertices
toric symmetry	:	$U(1) \times U(1)$	$U(1)$	-
gauge enhancement	:	$G_r \times U^2(1)$	$G_{r+1} \times U(1)$	G_{r+2}

In the present paper, we focus our attention mainly on the study of the typical family of base surfaces S of eq(1.1) involving the non planar complex tetrahedral surface and its blow ups denoted here below as $\mathcal{T}_0$ and $\mathcal{T}_n$ respectively. In the conclusion section, we give comments on the engineering of GUT-like 4D $\mathcal{N} = 1$ supersymmetric quiver gauge models based on $\mathcal{T}_0$ and $\mathcal{T}_n$. A more involved and explicit study for the engineering of F- theory GUT-like models along the line of the BHV approach; but now with $\mathcal{T}_0$ and $\mathcal{T}_n$ as complex base geometries in the local Calabi-Yau four-folds of eq(1.1) will be reported in [20].

The presentation of this paper is as follows: In section 2, we review general aspects of del Pezzo surfaces dP_k; in particular their 2- cycle homology classes and their links to the exceptional[1] Lie algebras. This review on real 2- cycle homology of the dP$_k$s is important to shed more light for the study and the building of the blow ups of the tetrahedron. In section 3, we introduce the complex tetrahedral surface $\mathcal{T}_0$; first as a complexification of the usual real tetrahedron (hollow triangular pyramid); that is as a non planar complex surface given by the intersection of four projective planes CP^2. Second as a complex codimension one divisor ("a toric boundary") of the complex three dimension projective space CP^3. We take also this opportunity to recall useful results on CP^3 thought of as a toric manifold and its Chern classes $c_k\left(TCP^3\right)$. These tools are used in section 4 to study the blow ups of the tetrahedron; in particular the toric blow ups of its vertices by projective planes and the blow up of its edges by the del Pezzo surface dP$_1$. In section 5, we give a conclusion and make comments on supersymmetric GUT-like quiver gauge theories embedded in F-theory compactification on local Calabi-Yau four-folds.

2. Del Pezzo surfaces dP_k

We first consider the 2- cycle homology of the del Pezzo surfaces. Then we give the links between these surfaces and the roots system of the "exceptional" Lie algebras.

2.1. Homology of dP_k

The dP_k del Pezzo surfaces with $k \leq 8$ are defined as blow ups of the complex projective space CP^2 at k points. Taking into account the overall size r_0 of the compact CP^2, a surface dP_k has then real $(k+1)$ dimensional Kahler moduli $\langle r_0, r_1, \ldots, r_k\rangle$ corresponding to the volume of each

[1] Here E_3, E_4, E_5 denote respectively $SU(3) \times SU(2)$, $SU(5)$ and $SO(10)$ and E_6, E_7, E_8 are the usual exceptional Lie algebras in Cartan classification.

blown up cycle. The dP_ks possess as well a moduli space of complex structures with complex dimension $(2k-8)$ where the eight gauge fixed parameters are associated with the $GL(3)$ symmetry of CP^2. As such, only surfaces with $5 \leq k \leq 8$ admit a moduli space of complex structures. The real 2-cycle homology group $\mathbb{H}_2(dP_k, Z)$ is $(k+1)$ dimensional and is generated by $\{H, E_1, ..., E_k\}$ where H denotes the hyperplane class inherited from CP^2 and the E_i denote the exceptional divisors associated with the blow ups. These generators have the intersection pairing

$$H.H = 1 \quad , \quad H.E_i = 0 \quad , \quad E_i.E_j = -\delta_{ij} \quad , \quad i,j = 1,...,k \quad , \tag{2.1}$$

so that the signature η of the $\mathbb{H}_2(dP_k, Z)$ group is given by $diag(+ - ... -)$.
The first three blow ups giving dP_1, dP_2 and dP_3 complex surfaces are of toric types while the remaining five others namely $dP_4, ..., dP_8$ are non toric. These projective surfaces have the typical toric fibration

$$\begin{array}{ccl} T^2 & \to & dP_k \\ & & \downarrow \pi_B \quad , \qquad k = 1,2,3, \\ & & \boldsymbol{B}_{2,k} \end{array} \tag{2.2}$$

with real real two dimension base $\boldsymbol{B}_{2,k}$ nicely represented by toric diagrams $\Delta_{2,k}$ encoding the toric data on the shrinking cycles in the toric fibration

surface S	dP_0= CP^2	dP_1	dP_2	dP_3
blow ups	$k=0$	$k=1$	$k=2$	$k=3$
toric graph $\Delta_{2,k}$	triangle	quadrilateral	pentagon	hexagon
generators	H	H , E_1	H , E_1 , E_2	H , E_1 , E_2 , E_2

The toric graphs of the projective plane CP^2 and its toric blown ups namely dP_1, dP_2 and dP_3 are depicted in the figure (1). The surfaces dP_k with $4 \leq k \leq 8$ have no toric graph representation.

In terms of the basic hyperline H and the exceptional curves E_i, generic classes $[\Sigma_a]$ of complex holomorphic curves in the del Pezzos dP_k are given by the following integral linear combinations,

$$\Sigma_a = n_a H - \sum_{i=1}^{k} m_{ai} E_i, \tag{2.3}$$

with n_a and m_a are integers. The self- intersection numbers $\Sigma_a^2 \equiv \Sigma_a \cdot \Sigma_a$ following from eqs(2.3) and (2.1) are then given by

$$\Sigma_a^2 = n_a^2 - \sum_{i=1}^{k} m_{ai}^2. \tag{2.4}$$

The canonical class Ω_k of the projective dP_k surface, which is given by *minus* the first Chern class $c_1(dP_k)$ of the tangent bundle of the surface dP_k, reads as,

$$\Omega_k = -\left(3H - \sum_{i=1}^{k} E_i\right), \tag{2.5}$$

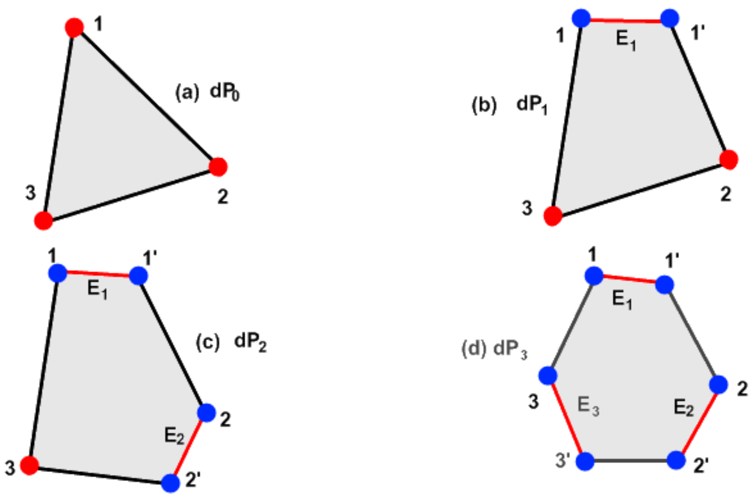


Figure 1: Toric graphs for dP_0 , dP_1 , dP_2 and dP_3. The surface dP_1 is obtained by blowing up the vertex 1. The other are recovered by blowing up the vertices 2 and 3.

and has a self intersection number $\Omega_k^2 = 9 - k$ whose positivity requires $k < 9$. Obviously $k = 0$ corresponds just to the case where there is no blow up. The degree d_Σ of a generic complex curve class $\Sigma = nH - \sum_{i=1}^{k} m_i E_i$ in dP_k is given by the intersection number between the class Σ with the anticanonical class $(-\Omega_k)$,

$$d_\Sigma = - \ (\Sigma \cdot \Omega_k) = 3n - \sum_{i=1}^{k} m_i. \tag{2.6}$$

Positivity of this integer d_Σ puts a constraint equation on the allowed values of the n and m_i integers which should be like,

$$\sum_{i=1}^{k} m_i \leq 3n. \tag{2.7}$$

Notice that there is a remarkable relation between the self intersection number Σ^2 (2.4) of the classes of holomorphic curves and their degrees d_Σ. This relation, which is known as the *adjunction formula* [21, 16], is given by $\Sigma^2 = 2g - 2 + d_\Sigma$, and allows to define the genus g of the curve class Σ as

$$g = 1 + \frac{n\,(n-3)}{2} - \sum_{i=1}^{k} \frac{m_i\,(m_i - 1)}{2}. \tag{2.8}$$

For instance, taking $\Sigma = 3H$; that is $n = 3$ and $m_i = 0$, then the genus g_{3H} of this curve is equal to 1 and so the curve $3H$ is in the same class of the real 2- torus. In general, fixing the genus g to a given positive integer puts then a second constraint equation on n and m_i integers; the first constraint is as in (2.7). For the example of rational curves with $g = 0$, we have $\Sigma^2 = d_\Sigma - 2$ giving a relation between the degree d_Σ of the curve Σ and its self intersection. For $d_\Sigma = 0$, we have a rational curve with self intersection $\Sigma^2 = -2$ while for $d_\Sigma = 1$ we have a self intersection $\Sigma^2 = -1$. To get the general expression of genus $g = 0$ curves, one has to solve the constraint equation $\sum_{i=1}^{k} m_i (m_i - 1) = 2 + n (n - 3)$ by taking into account the condition (2.7). For $k = 1$, this relation reduces to $m (m - 1) = 2 + n (n - 3)$, its leading solutions $n = 1, m = 0$ and $n = 0$, $m = -1$ give just the classes H and E respectively with degrees $d_H = 3$ and $d_E = 1$. Typical solutions for this constraint equation are given by the generic class $\Sigma_{n,n-1} = nH - (n-1) E$ which is more convenient to rewrite it as follows $\Sigma_{n,n-1} = H + (n-1)(H - E)$.

2.2. Link to roots of Lie algebras

Del Pezzo surfaces dP_k have also a remarkable link with the exceptional Lie algebras. Decomposing the $\mathbb{H}_2$ homology group like,

$$\mathbb{H}_2 (dP_k, Z)_{k\geq 3} \quad = \quad \langle \Omega_k \rangle \oplus \mathcal{L}_k \quad , \tag{2.9}$$

with

$$\begin{array}{lll} \Omega_k & = & -3H + E_i + \cdots + E_k \quad , \\ \mathcal{L}_k & = & \langle \Omega_k \rangle^{\perp} \quad , \end{array} \tag{2.10}$$

where the sublattice $\mathcal{L}_k = \langle \alpha_1, ..., \alpha_k \rangle$, which is orthogonal to Ω_k, is identified with the root space of the corresponding Lie algebra E_k. The generators α_i of the lattice $\mathcal{L}_k$ are:

$$\begin{array}{ll} \alpha_1 & = E_1 - E_2, \\ \vdots & \quad \vdots \\ \alpha_{k-1} & = E_{k-1} - E_k, \\ \alpha_k & = H - E_1 - E_2 - E_3, \end{array} \tag{2.11}$$

with pairing product $\alpha_i.\alpha_j$ equal to minus the Cartan matrix $C_{ij} (E_k)$ of the Lie algebra E_k. For the particular case of dP_2, the corresponding Lie algebra is $su(2)$. The mapping between the exceptional curves and the roots of the exceptional Lie algebras is given in the following table

dP_k surfaces	exceptional curves	Lie algebras	simple roots
dP_1	E_1	-	-
dP_2	E_1, E_2	$su(2)$	α_1
dP_3	E_1, E_2, E_3	$su(3) \times su(2)$	$\alpha_1, \alpha_2, \alpha_3$
dP_4	E_1, E_2, E_3, E_4	$su(5)$	$\alpha_1, \alpha_2, \alpha_3, \alpha_4$
dP_5	E_1, E_2, E_3, E_4, E_5	$so(10)$	$\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$
dP_6, dP_7, dP_8	$E_1,\ E_2,\ ...,\ E_k$	E_6, E_7, E_8	$\alpha_1, ..., \alpha_k, k = 6, 7, 8$

(2.12)

Notice that one can also use eqs(2.9,2.11) to express the generators H and $\langle E_i \rangle_{1\leq i\leq k}$ in terms of the anticanonical class Ω_k and the roots of the exceptional Lie algebra; for details see [20].

3. Tetrahedral surface

The complex tetrahedral surface $\mathcal{T}_0$ has much to do with the usual real triangular hollow[2] pyramid which we denote as $\Delta_{\mathcal{T}_0}$. In this section, we want to exhibit explicitly this link; but also its relation to the complex three dimension projective space CP^3. To that purpose, we first describe the relation between the complex tetrahedral surface $\mathcal{T}_0$ and the complex projective plane CP^2. Then we examine its relation with the complex three dimension space CP^3. Because of the link between $\mathcal{T}_0$ and CP^3, we take this occasion to give useful results on the homology of CP^3 which we use in section 4 to study the blowing up of the tetrahedron.

3.1. Link between $\mathcal{T}_0$ and CP^2

Roughly, the complex tetrahedral surface $\mathcal{T}_0$ extends the complex projective plane CP^2; it is a non planar projective surface that involve several projective planes $\{CP_a^2\}$ and whose basic properties may be read from the real tetrahedron $\Delta_{\mathcal{T}_0}$. The latter is given by the four external faces of the triangular hollow pyramid $\Delta_{\mathcal{T}_0}$ whose graph is depicted in (2).

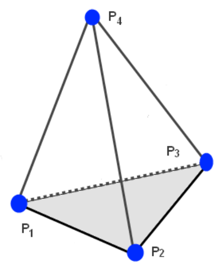


Figure 2: This figure represents the toric graph $\Delta_{\mathcal{T}_0}$ of the complex tetrahedral surface $\mathcal{T}_0$. This toric surface is a candidate for a base surface of local CY4s in the BHV theory. On the faces of $\Delta_{\mathcal{T}_0}$ live fields in adjoint representation of $G_r \times U^2\,(1)$, while on the edges lives bi- fundamentals and at vertices it lives tri- fields Yukawa couplings

Form the figures (1) and (2) as well as the relation between triangles[3] and projective planes, we immediately learn that there is a strong link between the usual tetrahedron $\Delta_{\mathcal{T}_0}$ and the complex tetrahedral surface $\mathcal{T}_0$. This non planar surface is then built in terms of four intersecting compact projective planes CP_1^2, CP_2^2, CP_3^2 and CP_4^2 which are in one to one correspondence with the four faces of $\Delta_{\mathcal{T}_0}$. The intersection of any two projective planes; say CP_a^2 and CP_b^2, is a complex

[2]One should distinguish two kinds of triangular pyramids: filled and empty. We are interested in the second one denoted as $\Delta_{\mathcal{T}_0}$. The real triangular pyramid with filled bulk is denoted by Δ_{CP^3} ; it is the toric graph of CP^3. We also have the relation $\Delta_{\mathcal{T}_0} = \partial\,(\Delta_{CP^3})$.

[3]In toric geometry, projective lines CP^1 are presented by segments $[AB]$, projective planes CP^2 by triangles $[ABC]$ and in general CP^n spaces by n-simplex $[A_1...A_{n+1}]$.

projective line $\Sigma_{(ab)} \sim CP^1$ and are associated with the edges of $\Delta_{\mathcal{T}_0}$;

$$\begin{array}{ll} \Sigma_{(ab)} & = CP_a^2 \cap CP_b^2 \quad , \\ \Sigma_{(ab)} & \simeq CP^1 \quad , \end{array} \tag{3.1}$$

with $b > a = 1, ..., 4$. Moreover we learn also that any triplet of three projective planes; say CP_a^2, CP_b^2 and CP_c^2, meet at one of the four vertices of the tetrahedron, i.e:

$$P_{(abc)} \quad = \quad CP_a^2 \cap CP_b^2 \cap CP_c^2 \quad . \tag{3.2}$$

Up on using eq(3.1) may be also written as

$$\begin{array}{lll} P_{(abc)} & = & \Sigma_{(ab)} \cap CP_c^2 \quad , \\ & = & \Sigma_{(bc)} \cap CP_a^2 \quad , \\ & = & \Sigma_{(ac)} \cap CP_b^2 \quad , \end{array} \tag{3.3}$$

with $c > b > a = 1, ..., 4$. These vertices may be as well defined as the intersection of edges $\Sigma_{(ab)}$ and $\Sigma_{(bc)}$ or equivalently $\Sigma_{(bc)}$ and $\Sigma_{(ac)}$.
Notice that the exact link between $\mathcal{T}_0$ and $\Delta_{\mathcal{T}_0}$ is given by toric geometry which allows to define the complex tetrahedral surface $\mathcal{T}_0$ in terms of the following toric fibration,

$$\begin{array}{ccc} T^2_{\mathcal{T}_0} & \rightarrow & \mathcal{T}_0 \\ & & \downarrow \pi \\ & & B_{\mathcal{T}_0} \end{array} \tag{3.4}$$

where the fiber T^2 stands for the 2- torus $S^1 \times S^1$ and $B_{\mathcal{T}_0}$ for a real two dimensional base. The polytope $\Delta_{\mathcal{T}_0}$ is precisely the toric graph of the real base $B_{\mathcal{T}_0}$. This toric graph encodes the toric data of the toric symmetries of the complex tetrahedral surface viewed as a complex two dimension toric manifold. As these toric data are intimately related to the toric representation of CP^3; we give these details in the next section.

3.2. Relation between $\mathcal{T}_0$ and CP^3

Along with its connection with CP^2, the complex tetrahedral surface $\mathcal{T}_0$ has as well a strong link with the complex three dimension projective space CP^3. The projective planes CP_1^2, CP_2^2, CP_3^2 and CP_4^2 encountered in the previous subsection are precisely the four basic divisors D_1, $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_4$ of CP^3. In terms of the holomorphic coordinates $\{x_a\}$ of the complex four dimension space C^4 where live CP^3, we can define these basic divisors $\mathcal{D}_a$ by the following hypersurfaces,

$$\mathcal{D}_a = \left\{ \begin{array}{c} (x_1, x_2, x_3, x_4) \equiv (\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4) \\ (x_1, x_2, x_3, x_4) \neq (0, 0, 0, 0) \text{ and } x_a = 0 \end{array} \right\} \tag{3.5}$$

with $a = 1, 2, 3, 4$ and where λ is a non zero complex number; the parameter of the C^* action. In this set up, the complex tetrahedral surface may be defined as the complex codimension one hypersurface

$$\mathcal{T}_0 \quad = \quad \bigcup_{a=1}^{4} \mathcal{D}_a, \quad , \tag{3.6}$$

together with the following bi- and tri- intersections

$$\begin{array}{lll} \Sigma_{(ab)} & = & \mathcal{D}_a \cap \mathcal{D}_b \quad , \quad a < b \quad , \\ P_{(abc)} & = & \mathcal{D}_a \cap \mathcal{D}_b \cap \mathcal{D}_c \quad , \quad a < b < c \quad . \end{array} \tag{3.7}$$

Notice in passing that in the effective 4D space time physics of branes wrapping cycles in type II strings on Calabi Yau threefolds and F-theory on CY4-folds, these cycles intersections give rise to branes intersections which have a nice interpretation in terms of chiral matter in bi-fundamentals and tri-fields couplings.
In the toric geometry language, the complex tetrahedral surface $\mathcal{T}_0$ is in some sens[4] the "toric boundary" of CP^3. Recall that CP^3 is a toric manifold with the toric fibration

$$\begin{array}{ccc} T^3 & \rightarrow & CP^3 \\ & & \downarrow \pi \\ & & B_3 \end{array} \tag{3.8}$$

where the real three dimension base B_3 has as a toric polytope given by the 3- simplex Δ_{CP^3}. This 3- simplex is just the triangular pyramid with filled bulk and is related to $\Delta_{\mathcal{T}_0}$ as follows,

$$\Delta_{\mathcal{T}_0} = \partial \left(\Delta_{CP^3} \right) . \tag{3.9}$$

As such the complex tetrahedral surface $\mathcal{T}_0$ inherits specific features of the toric data of the complex projective space CP^3. These toric data, which are encoded on the faces, the edges and the vertices of the polytope Δ_{CP^3}, are generated by shrinking cycles of the T^3 fiber of eq(3.8). In the next section we will use these data to study the toric blown ups of $\mathcal{T}_0$; but before that let us complete this discussion by recalling useful results on the Chern classes for CP^3. These classes may be read from the total Chern class given by the following sum

$$c_{tot}\left(X\right) = 1 + c_1\left(X\right) + c_2\left(X\right) + c_3\left(X\right) , \tag{3.10}$$

where X stands for complex three dimension manifold and where the $c_k\left(X\right)$ refer to $c_k\left(TX\right)$; i.e the k-th Chern class of the tangent bundle TX.
For the case $X = CP^3$, the Chern classes $c_k\left(X\right)$ are generated by a single two dimensional class ω reads as follows

$$c_{tot}\left(X\right) \quad = \quad \left(1+\omega\right)^4 = 1 + 4\omega + 6\omega^2 + 4\omega^3 \quad , \tag{3.11}$$

together with the normalization

$$\int_{CP^3} \omega^3 = 1 \tag{3.12}$$

and the nilpotent relation $\omega^4 = 0$. From the relations (3.10) and (3.11), we can read directly the expression of the first $c_1\left(X\right)$, the second $c_2\left(X\right)$ and the third $c_3\left(X\right)$ Chern classes,

$$\begin{array}{lll} c_1\left(X\right) & = & 4\omega \quad , \\ c_2\left(X\right) & = & 6\omega^2 \quad , \\ c_3\left(X\right) & = & 4\omega^3 \quad , \end{array} \tag{3.13}$$

[4]What we mean by the toric boundary of a complex n dimension manifold M_n is the codimension one toric submanifold $M_{n-1} = \partial\left(M_n\right)_{toric}$ associated with the shrinking of then n-torus fiber T^n of M_n down to T^{n-1}.

as well as the Euler characteristic

$$\chi(X) = \int_{CP^3} c_3(X) = 4 \tag{3.14}$$

in agreement with the Gauss Bonnet theorem for CP^3. Notice that expressing the normalization condition (3.12) like

$$\int_{CP^3} \omega \wedge \omega^2 = 1, \tag{3.15}$$

one learns amongst others that that real 2- forms and real 4- forms are dual in CP^3. The same duality is valid for real 2- cycles Σ and codimension 2 real 4-cycles D that satisfy the following pairings:

$$\begin{array}{lcl} \langle \Sigma, D \rangle_{CP^3} & = & 1 \quad , \\ \int_D \omega^2 & = & 1 \quad , \\ \int_\Sigma \omega & = & 1 \quad , \end{array} \tag{3.16}$$

Notice moreover that the 2-form ω is the curvature of a line bundle $\mathcal{L}^*$ whose complex conjugate $\mathcal{L}$ is precisely the generating line bundle over CP^3 with total Chern class,

$$c_{tot}(\mathcal{L}) = 1 - \omega. \tag{3.17}$$

A remarkable line bundle over CP^3 is given by the maximum exterior power of the cotangent bundle T^*X with $X = CP^3$. This is the canonical line bundle

$$\mathcal{K} = (T^*X) \wedge (T^*X) \wedge (T^*X) \tag{3.18}$$

whose Chern class given by $c_{tot}(\mathcal{K}) = 1 - 4\omega$ with $c_1(\mathcal{K}) = -4\omega$. From these relations, we learn that $\mathcal{K}$ is the fourth power of the generating line bundle $\mathcal{L}$,

$$\mathcal{K} = \mathcal{L}^4 \tag{3.19}$$

We learn as well that $c_1(\mathcal{K})$ is nothing but $c_1(T^*X) = -c_1(TX)$.

4. Blown up geometries

First notice that the blow ups of the tetrahedral surface may be classified in two types: toric blow ups and non toric ones. In this section, we will mainly focus on the toric blow ups which can be engineered directly from the toric graph $\Delta_{\mathcal{T}_0}$ given by the figure (2).

Moreover, within the class of toric blow ups, we also distinguish two subsets of toric blow ups of $\mathcal{T}_0$ depending on the dimension of the shrinking cycles:

(**1**) blow ups of the four vertices of $\Delta_{\mathcal{T}_0}$ in terms of projective planes CP^2. These are the analogs of the blow ups we encounter in building del Pezzo surfaces form the projective plane. They are associated with singularities at isolated points.

(**2**) blow ups the edges of $\Delta_{\mathcal{T}_0}$ by using projective lines CP^1. This kind of blow ups has no analog

in the blowing up of CP^2. The blown ups surfaces will be denoted as $\mathcal{T}_k'$.
Recall that at the four vertices of the tetrahedron Δ_{CP^3}, a 3-torus T^3 shrinks to zero

$$vertex \quad : \quad T^3 \quad \rightarrow \quad 0 \quad , \tag{4.1}$$

while on its six the edges we have shrinking 2-tori.

$$edge \quad : \quad T^2 \quad \rightarrow \quad 0 \quad . \tag{4.2}$$

Below, we study these two kinds of blow ups by first considering blowing ups by projective planes by essentially mimicking the building of del Pezzo surfaces in terms of the blow ups of CP^2 considered in section 2.

4.1. Blow ups of points by $\mathbf{CP^2}$s

To start notice that by thinking about the complex tetrahedral surface $\mathcal{T}_0 \sim T^2 \times \Delta_{\mathcal{T}_0}$ as a toric submanifold of $CP^3 \sim T^3 \times \Delta_{CP^3}$, that is roughly as its toric boundary; see also footnote (2),

$$\Delta_{\mathcal{T}_0} = \partial \left(\Delta_{CP^3}\right) \quad , \quad \mathcal{T}_0 \sim \partial \left(CP^3\right)_{toric} \quad , \tag{4.3}$$

one can construct the leading terms of the family $\{\mathcal{T}_n\}$ of the blown ups of the complex tetrahedral surface just by help of the power of toric geometry. Indeed, using the toric relation $\Delta_{\mathcal{T}_0} = \partial \left(\Delta_{CP^3}\right)$, one sees that the toric action $U^3 \left(1\right)$ generated by translations on the fiber T^3 of CP^3 (3.8) has fix points associated with shrinking p- cycles in T^3. These are:
(**1**) the divisors of CP^3; in particular for the four basic $\mathcal{D}_a$ given by eqs(3.5). On these basic divisors, a 1-cycle of the 3-torus T^3 fibration in the bulk of Δ_{CP^3} shrinks to zero. As such one is left with T^2 fibers on the $\mathcal{D}_a$'s as well as a $U\left(1\right) \times U\left(1\right)$ toric action as a residual subsymmetry of the $U^3 \left(1\right)$ symmetry of the bulk geometry:

$$\begin{array}{lcc} CP^3 & : & \text{basic divisors } \mathcal{D}_a \\ T^3 & \rightarrow & T^2 \\ U^3\left(1\right) & \rightarrow & U\left(1\right) \times U\left(1\right) \end{array} \tag{4.4}$$

(**2**) the edges $\Sigma_{(ab)}$ of the tetrahedron on which 2-cycles of T^3 shrink to zero. Recall that these edges, which are described by projective lines, are given by the following intersections,

$$\Sigma_{(ab)} \quad = \quad \mathcal{D}_a \cap \mathcal{D}_b \quad . \tag{4.5}$$

Being toric submanifolds; the complex codimension one divisors $\mathcal{D}_a$ have as well a toric fibration which we write as follows:

$$\begin{array}{ccc} T_a^2 & \rightarrow & D_a \\ & & \downarrow \pi_a \\ & & \Delta_{D_a} \end{array} \tag{4.6}$$

where the toric polytope describing Δ_{D_a} is a triangle. Similarly, the intersecting curve $\Sigma_{(ab)}$ of the two divisors D_a and D_b is also toric with the typical fibration

$$\begin{array}{ccc} S^1_{(ab)} & \rightarrow & \Sigma_{(ab)} \\ & & \downarrow \pi_{(ab)} \\ & & \Delta_{\Sigma_{(ab)}} \end{array} \tag{4.7}$$

where now $\Delta_{\Sigma_{(ab)}}$ is represented by a segment of a straight line. As such, along the curves $\Sigma_{(ab)}$ the bulk 3- cycles of T^3 shrinks down to a 1- cycle fibers $S^1_{(ab)}$ fibers and the $U^3(1)$ bulk toric action gets reduced to $U(1)$.

$$\begin{array}{ccc} CP^3 & & \text{edges } \Sigma_{(ab)} \\ T^3 & \rightarrow & S^1_{(ab)} \\ U^3(1) & \rightarrow & U_{(ab)}(1) \end{array} \tag{4.8}$$

At each point of these projective lines $\Sigma_{(ab)}$ lives then an ordinary A_1 type singularity associated with the shrinking of T^2 whose blow up is done in terms of a real two sphere.
(**3**) the vertices $P_{(abc)}$ of the tetrahedron given by the tri- intersection,

$$P_{(abc)} = \mathcal{D}_a \cap \mathcal{D}_b \cap \mathcal{D}_c \quad . \tag{4.9}$$

At these four vertices, the 3-cycle T^3 in the bulk geometry shrinks completely to zero and one is left with a larger singularity involving three intersecting ordinary A_1 type singularities which might be thought of as the affine A_2 type singularity depicted in the figure (3). Using the toric fibrations (4.6) of the basic divisors $\mathcal{D}_a$, we clearly see that each ordinary A_1 singularity is associated with the shrinking of the T^2_a torus at the tri- vertex intersection (4.9).

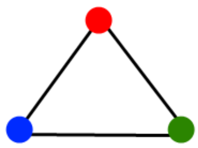


Figure 3: The Dynkin diagram of the affine A_2 singularity. Viewed as the intersecting of three divisors $D_i \sim T^2_i \times \Delta_i$, the singularity at the vertex $P_{abc} = D_a \cap D_b \cap D_c$ involves the simultaneous shrinking of the three T^2_is to zero. Each node is associated with the shrinking of one of the T^2_is interpreted as an ordinary A_1 singularity.

With these features on the toric projective space CP^3 and their links to the toric tetrahedral surface $\mathcal{T}_0$ in mind, we turn now to study the toric blow ups of the tetrahedron.

4.1.1. Blow ups of CP[3]

By mimicking the analysis of section 2 regarding the construction of the *eight* del Pezzo surfaces dP_n from the projective plane CP^2 and using group theoretical arguments[5], one learns that we may a priori perform up to fifteen blow ups of points in CP^3 by projective planes. In these blow ups, the fifteen points which we denote as

$$P_1, ..., P_{15} \quad \in \quad CP^3 \quad , \tag{4.10}$$

get replaced by exceptional projective planes $\digamma_i$; $i = 1, ..., 15$. Because the complex dimension of CP^3 is odd, we don't have a self dual homological mid- class and so the derivation of the number *15* need a little bit more work than in the CP^2 case. A way to get this number is to compute the pairing product $\langle \Omega_k \Omega_k^* \rangle$ of the following real 4- cycle Ω_k and its dual 2- cycle Ω_k^*,

$$\begin{array}{lll} \Omega_k & = & 4G - \sum_{i=1}^k \digamma_i \quad , \\ \Omega_k^* & = & 4H - \sum_{i=1}^k E_i \quad . \end{array} \tag{4.11}$$

In these relations G is a hyperplane in CP^3 and $\digamma_i$ are the generators of the blow ups. The generators $H \equiv G^*$ and $E_i \equiv \digamma_i^*$ are respectively the dual classes of G and $\digamma_i$ satisfying the following pairing products

$$\begin{array}{llllll} \langle H, G \rangle & = 1 & , & \langle E_i, \digamma_j \rangle & = -\delta_{ij} & , \\ \langle G, G \rangle & = 0 & , & \langle E_i, F_j \rangle & = 0 & , \\ \langle H, H \rangle & = 0 & , & \langle \digamma_i, \digamma_j \rangle & = 0 & . \end{array} \tag{4.12}$$

Using these relations, we can compute the product $\langle \Omega_k \Omega_k^* \rangle$ in terms of the positive integer k. We find

$$\langle \Omega_k \Omega_k^* \rangle = 16 - k. \tag{4.13}$$

Positivity of this pairing product requires that the integer k should be less than 16. From this result, we learn that the complex tetrahedral surface $\mathcal{T}_0$ has a family $\{\mathcal{T}_k\}$ of *fifteen* cousins

$$\mathcal{T}_1 \quad , \quad \dots \quad , \quad \mathcal{T}_{15} \quad , \tag{4.14}$$

obtained by blown ups of isolated points of $\mathcal{T}_0$ by projective planes CP^2. We will see later that the complex tetrahedral surface $\mathcal{T}_0$ has a second family $\{\mathcal{T}_m'\}$ of *thirty five* cousins. But before coming to that notice the complex codimension one divisor $\mathcal{T}_0$ of the complex projective space CP^3 is described by the real 4- cycles

$$\Omega_0 = 4G, \tag{4.15}$$

where, in toric language, the number *4* in above relation refers to the four basic divisors $\mathcal{D}_a$. Similarly, we have the dual class $\Omega_0^* = 4H$ associated with the classes of complex lines normal to the class of the complex surfaces $\mathcal{D}_a$ of the complex three dimension space CP^3.

[5]Notice that $CP^2 \subset C^3$ with dimension of the structure group as $\dim SU(3) = 8$. We also have $CP^3 \subset C^4$ with $\dim SU(4) = 15$.

Regarding the second family $\{\mathcal{T}'_m\}$ of cousins of the complex tetrahedral surface $\mathcal{T}_0$, notice that along with the divisor class Ω_0 given by eq(4.15), we may also define the 2- cycle class Υ_0 associated with the six edges $\Sigma_{(ab)}$ of the tetrahedron,

$$\Upsilon_0 = 6H. \tag{4.16}$$

Its dual class is given by the real 4- cycle $\Upsilon_0^* = 6G$ and it describes the class of the six complex surfaces $\Gamma_{(ab)}$ in CP^3 that are normal to the edges $\Sigma_{(ab)}$. Moreover, using the exceptional curves E_i, one may define in general the following real 2- cycles class

$$\Upsilon_n = 6H - \sum_{i=1}^{n} E_i, \tag{4.17}$$

where a priori n is a positive integer. Computing the pairing $\langle \Upsilon_n \Upsilon_n^* \rangle$ where Υ_n^* stands for the dual 4-cycle class which reads in terms of the generators G and $\digamma_i$ like $\Upsilon_n^* = 6G - \sum_{i=1}^{n} \digamma_i$, we get

$$\langle \Upsilon_n \Upsilon_n^* \rangle = 36 - n, \tag{4.18}$$

whose positivity require that n should be less than 36. From this result, we learn that we may perform up to *35* blow ups by projective line in CP^3; these are precisely the second family of cousins of the complex tetrahedron

$$\mathcal{T}'_1 \quad , \quad \ldots \quad , \quad \mathcal{T}'_{35} \quad . \tag{4.19}$$

Furthermore, we may define as well generic real 4- cycles $[C_a]$ in the complex three dimension space CP^3. They are given by the following linear combination

$$C_a \quad = \quad n_a G - \textstyle\sum_i m_{ai} \digamma_i \quad , \tag{4.20}$$

with

$$\textstyle\int_G \omega^2 = +1 \quad , \quad \int_{\digamma_i} \omega^2 = -1 \quad , \tag{4.21}$$

and where the 2- form ω is as in eqs(3.11-3.17) and where n_a and m_{ai} are integers. By duality, we also have the real 2- cycles basis $[\Sigma_a] \equiv [C_a^*]$ in CP^3 which are given by the following linear combination

$$\Sigma_a \quad = \quad n_a H - \textstyle\sum_i m_{ai} E_i \quad , \tag{4.22}$$

with

$$\textstyle\int_H \omega = +1 \quad , \quad \int_{E_i} \omega = -1 \quad .$$

The real 2-cycles Σ_a are in some sense the normal to the real 4-cycles C_a in the complex space CP^3. Their intersection is given by the pairing product; we have:

$$\langle C_a \Sigma_b \rangle = n_a n_b - \sum_i m_{ai} m_{bi}. \tag{4.23}$$

We end these comments by recalling that alike in the blowing up of CP^2, here also we have the two kinds of blow ups: toric and non toric. In what follows, we consider the toric blow ups concerning the blowing up of the vertices and the edges of the tetrahedron. The blowing of the vertices will be done in terms of projective planes while those of the edges will be done in terms of projective line.

4.1.2. Blowing up the vertices

One blow up: geometry $\mathcal{T}_1$
The blow up of one of the four vertices of the tetrahedron $\Delta_{\mathcal{T}_0}$; say the fourth vertex P_4 in the diagram (2); is depicted in figure (4).

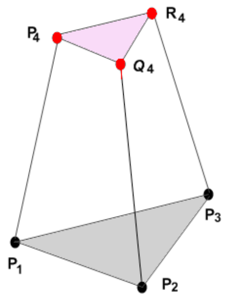


Figure 4: Surface $\mathcal{T}_1$; it is given by the blow up of the vertex P_4 of the tetrahedron by a projective plane CP^2 which is described by the triangle $[P_4Q_4R_4]$.

In toric language, the blow up of the point P_4 by a projective plane amounts to replace the vertex P_4 of the tetrahedron by a triangle $[P_4Q_4R_4]$, that is making the substitution

$$P_4 \qquad \rightarrow \qquad [P_4Q_4R_4]\,. \tag{4.24}$$

As a consequence the tetrahedron $\Delta_{\mathcal{T}_0}$ gets deformed to a complex geometry with toric graph $\Delta_{\mathcal{T}_1}$ having five faces. These faces are as follows:
(**i**) two triangles

$$[P_1P_2P_3] \quad , \quad [P_4Q_4R_4] \quad , \tag{4.25}$$

representing two non intersecting projective planes CP_1^2 and CP_2^2. This non intersecting property of the two projective planes is easily read in the toric geometry language by determining the intersection of the above triangles:

$$[P_1P_2P_3] \cap [P_4Q_4R_4] \quad = \quad \emptyset \quad . \tag{4.26}$$

(**ii**) three quadrilaterals

$$[P_1P_2P_4Q_4] \quad , \quad [P_1P_3P_4R_4] \quad , \quad [P_2P_3Q_4R_4] \quad , \tag{4.27}$$

describing three intersecting del Pezzo surfaces $dP_1^{(1)}$, $dP_1^{(2)}$ and $dP_1^{(3)}$. These intersections may be directly read from the polytope $\Delta_{\mathcal{T}_1}$ of the figure (4). We have:

$$\begin{array}{lcl} [P_1P_4] &=& [P_1P_2P_4Q_4] \cap [P_1P_3P_4R_4] \quad , \\ [P_2Q_4] &=& [P_1P_2P_4Q_4] \cap [P_2P_3Q_4R_4] \quad , \\ [P_3R_4] &=& [P_1P_3P_4R_4] \cap [P_2P_3Q_4R_4] \quad , \end{array} \tag{4.28}$$

Notice that using the generator G and $\digamma_i$, we can define the blow up surface represented by the polytope $\Delta_{\mathcal{T}_1}$ in terms of the "canonical 4- cycle" as follows:

$$\Omega_1 = 4G - \digamma_1, \tag{4.29}$$

where $\digamma_1$ generates the blow up (4.24). Notice also the emergence of the del Pezzo surfaces dP_1 into the geometry of the blown up of the complex tetrahedral surface. This result is not a strange thing since it was expected from the analysis of section 2 since after all the blown up of the tetrahedron involves implicitly the blowing up of projective planes constituting the tetrahedral surface.

Two blow ups: geometry $\mathcal{T}_2$

In the case of the blown up of two vertices of the tetrahedron, say the third vertex P_3 and the fourth P_4 one, we get a geometry $\mathcal{T}_2$ that involves more intersecting del Pezzo surfaces. The toric graph $\Delta_{\mathcal{T}_2}$ of this blown up surface is depicted in figure (5),

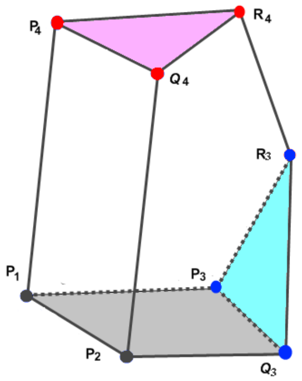


Figure 5: Surface $\mathcal{T}_2$; it is given by the blow up of the vertices P_3 and P_4 of the tetrahedron by two respective projective planes CP_1^2 and CP_2^2 described by the triangles $[P_3Q_3R_3]$ and $[P_4Q_4R_4]$.

The corresponding polytope $\Delta_{\mathcal{T}_2}$ has six intersecting faces as reported in the following table,

$$\begin{array}{lclclc} \text{triangles} & : & [P_3Q_3R_3] & , & [P_4Q_4R_4] & , \\ \text{quadrilaterals} & : & [P_1P_2P_4Q_4] & , & [P_1P_2P_3Q_3] & , \\ \text{pentagons} & : & [P_1P_3R_3R_4P_4] & , & [P_2Q_3R_3Q_4R_4] & , \end{array} \tag{4.30}$$

from which one may read directly the intersections. The triangles describe respectively two projective planes $dP_0^{(1)}$ and $dP_0^{(2)}$, the quadrilaterals describe two del Pezzo dP_1 surfaces defining the third and the fourth faces $dP_1^{(3)}$ and $dP_1^{(4)}$ and the pentagons are associated with two del Pezzo dP_2 geometries giving the fifth and the sixth faces $dP_2^{(5)}$ and $dP_2^{(6)}$.

Using the real 4- cycle generators G and $\digamma_i$, the real 4- cycle class $[\Omega_2]$ describing the two blow ups of the tetrahedron is given by

$$\Omega_2 = 4G - \digamma_1 - \digamma_2 , \tag{4.31}$$

where F_1 and F_2 generate the blown ups of the points

$$\begin{array}{lll} P_3 & \rightarrow & [P_3Q_3R_3] \quad , \\ P_4 & \rightarrow & [P_4Q_4R_4] \quad . \end{array} \tag{4.32}$$

$\mathcal{T}_3$ and $\mathcal{T}_4$ geometries

Similar analysis may be done for the blown up of three and four vertices. For the blown up of three vertices; say P_2, P_3 and P_4, one gets a polytope $\Delta_{\mathcal{T}_3}$ with seven intersecting faces: (i) three triangles, (ii) three pentagons and (iii) an hexagon,

$$\begin{array}{lllll} \text{triangles} & : & [P_2Q_2R_2] \quad , & [P_3Q_3R_3] \quad , & [P_4Q_4R_4] \quad , \\ \text{pentagons} & : & [P_1P_2R_2Q_4P_4] \quad , & [P_1P_3R_3R_4P_4] \quad , & [P_2Q_3R_3Q_4R_4] \quad , \\ \text{hexagon} & : & [R_2Q_2Q_3R_3Q_4R_4] \quad . & & \end{array} \tag{4.33}$$

For the blown up of the four vertices of the tetrahedron, one obtains the geometry depicted in the figure (6),

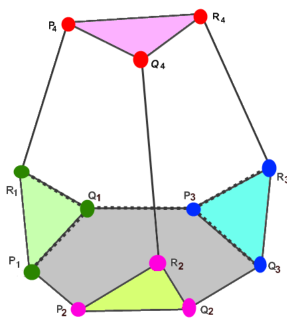


Figure 6: Geometry $\mathcal{T}_4$; its given by the blow up of the four vertices of the tetrahedron by a projective planes CP^2.

The resulting toric graph $\Delta_{\mathcal{T}_4}$ has twelve vertices and eight faces given by four triangles describing four exceptional projective planes and four hexagons associated with the del Pezzo surfaces dP_3.

4.2. Blow ups by $\mathbf{CP}^1$s

Here also we restrict our analysis to the toric blow ups concerning the A_1 singularities degenerating along the edges $\Sigma_{(ab)}$ eqs(4.5-4.7) of the tetrahedron $\Delta_{\mathcal{T}_0}$. To that purpose, notice first that contrary to the familiar cases where 2- cycle degeneracies takes place at isolated points on manifolds, here the A_1 singularity take place along the edges of the tetrahedron $\Delta_{\mathcal{T}_0}$; that is for all those points P of the tetrahedron $\Delta_{\mathcal{T}_0}$ where a 2- torus shrinks down to zero.

4.2.1. From a edge to a dP_1 surface

First recall that in the case of singularities at isolated points the blow up is achieved in terms a complex surface namely a projective plane CP^2. Here we complete this study by showing that for the case of the A_1 singularity on the edges, the blow up is achieved as well in terms of a complex surface but this time in terms of dP_1 surface.

Indeed, given a segment $[AB]$ describing a projective line CP^1 where at each point $P \in [AB]$ lives a A_1 singularity, the blow up of such singularity consists to replace each point P by a segment $[PQ]$ as it is usually done,

$$P \quad \to \quad [PQ] \quad . \tag{4.34}$$

This means that each singular point P is substituted by a rational curve. Doing so for all points P belonging to the segment $[AB]$, we end with the quadrilateral

$$[AB] \times [CD] \,. \tag{4.35}$$

The blowing up of an edge $[P_aP_b]$ of the tetrahedron $\Delta_{\mathcal{T}_0}$ of the figure (2) corresponds in the language of toric graphs to the replacement

$$[P_aP_b] \quad \to \quad P_aP_b \times [Q_aQ_b] \sim [P_aQ_aP_bQ_b] \quad . \tag{4.36}$$

In complex geometry, the blow up of an edge $\Sigma_{(ab)} \sim CP^1$ of the complex tetrahedral surface amounts to replace the complex projective line CP^1 by a del Pezzo surface dP_1:

$$CP^1 \quad \to \quad dP_1 \quad . \tag{4.37}$$

Let us apply this construction to the blowing up of two independent edges of the tetrahedron $\Delta_{\mathcal{T}_0}$; say $[P_1P_3]$ and $[P_2P_4]$ with $[P_1P_3] \cap [P_2P_4] = \emptyset$.

First, we study the blow up of the edge $[P_2P_4] \in \Delta_{\mathcal{T}_0}$ of the figure (2) and then we consider the blow up of the two edges $[P_1P_3]$ and $[P_2P_4]$.

4.2.2. Blowing up the edge $[P_2P_4]$

The blow up of the edge $[P_2P_4]$ of the tetrahedron of the graph (2) is depicted in the figure (7). The edge $[P_2P_4]$, which represents a complex projective line, has been replaced by the quadrilateral $[P_2Q_2P_4Q_4]$:

$$[P_2P_4] \quad \to \quad [P_2Q_2P_4Q_4] \quad . \tag{4.38}$$

The obtained polytope has five faces and six vertices where meet three faces as well as three edges. Regarding the five faces, we have:

(i) two triangles $[P_1P_2P_4]$ and $[Q_2P_3Q_4]$ describing two projective planes.

(ii) three quadrilaterals $[P_1P_2Q_2P_3]$, $[P_2Q_2P_4Q_4]$, and $[P_1P_3P_4Q_4]$ describing dP_1 surfaces. These del Pezzo surfaces intersects mutually and intersect as well with the projective planes.

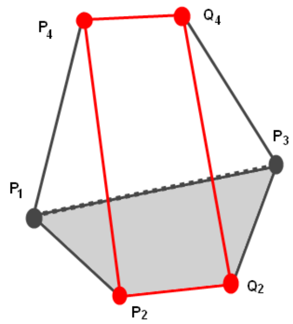


Figure 7: Surface $\mathcal{T}_1'$; it is given by the blow up of a edge $[P_2P_4]$ of the graph fig(2) by a projective planes CP^1. The resulting geometry is a del Pezzo surface described by the polygon $[P_2Q_2P_4Q_4]$. The full geometry has five faces del Pezzo surfaces whose intersections are directly read fom the toric graph.

4.2.3. Blowing up the edge $[P_1P_3]$ and $[P_2P_4]$

The blow up of two edges of the tetrahedron by projective lines is depicted in the figure (8). The edges $[P_1P_3]$ and $[P_2P_4]$ have been replaced by the quadrilaterals $[P_1Q_1P_3Q_3]$ and $[P_2Q_2P_4Q_4]$. The obtained polytope has six quadrilateral faces describing six intersecting del Pezzo surfaces dP_1.

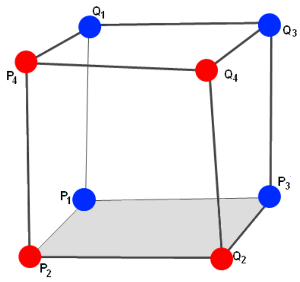


Figure 8: Surface $\mathcal{T}_2'$: it is given by the blow up of two edges of the tetrahedron fig(2): $[P_1P_3] \rightarrow [P_1Q_1P_3Q_3]$ and $[P_2P_4] \rightarrow [P_2Q_2P_4Q_4]$. The resulting geometry has eight vertices, twelve edges and six faces describing six intersecting del Pezzo surfaces dP_1s. The intersections are directly read on this graph.

5. Conclusion and discussions

Motivated by F-theory- GUT models building along the line of the BHV approach [11, 12, 13] and guided by special properties of the toric fibration of complex surfaces, we have studied in this paper two families of blowing up of the complex tetrahedral surfaces $\mathcal{T}_0$. These families, which were respectively denoted as $\mathcal{T}_n$ with $n \leq 15$ and $\mathcal{T}_k'$ with $k \leq 35$ are as follows:

1) the blowing up of the complex three dimension space CP^3 up to fifteen isolated points by projective planes CP^2. Four of these blow ups are of toric type and have been explicitly studied by using the power of the standard toric graph representation and n-simplex description. If denoting by

$$\left(CP^3\right)_{n,0} \quad , \quad n = 1, ..., 15 \quad , \tag{5.1}$$

the blowing ups of the CP^3 at n isolated points, then the link between these $\left(CP^3\right)_{n,0}$s and the blown up tetrahedral surfaces $\mathcal{T}_n$ is given by means of toric geometry where roughly the $\mathcal{T}_n$s appear as their toric boundary; see also footnote (2).
Notice that viewed from the CP^3 side, the toric singularity at the tetrahedron vertices $P_{(abc)}$ is given the shrinking of a real 3-torus of the fibration $CP^3 \sim T^3 \times \Delta_{CP^3}$. On the complex tetrahedral surface side however, the visible toric singularity at

$$P_{(abc)} = \mathcal{D}_a \cap \mathcal{D}_b \cap \mathcal{D}_c, \tag{5.2}$$

is given by simultaneous shrinking of three 2- tori namely the shrinking of the toric fibers T_a^2, T_b^2 and T_c^2 of the respective divisors $\mathcal{D}_a$, $\mathcal{D}_b$ and $\mathcal{D}_c$; see eq(4.6).
2) the blowing up of CP^3 up to thirty five projective lines. Six of these blow ups are of toric type. These blow ups are different from the $\left(CP^3\right)_{n,0}$ ones since they concern the blown up of A_1 singularities. We may refer to them as,

$$\left(CP^3\right)_{0,k} \quad , \quad k = 1, ..., 35 \quad . \tag{5.3}$$

in order to distinguish them of the previous $\left(CP^3\right)_{n,0}$ family. In this case, the toric singularity living on the tetrahedron edges

$$\Sigma_{(ab)} = \mathcal{D}_a \cap \mathcal{D}_b \tag{5.4}$$

is associated with the shrinking of a real 2-torus of the fibration $CP^3 \sim T^3 \times \Delta_{CP^3}$ down to S^1 as shown on eq(4.8). Viewed from the divisors $\mathcal{D}_a$ and $\mathcal{D}_b$, the singularity on the edge corresponds to the shrinking of a 1-cycle along the intersection of $\mathcal{D}_a$ and $\mathcal{D}_b$.
Through this study we learned a set of special features amongst which the two following:
a) the toric blown ups $\mathcal{T}_n$ and $\mathcal{T}_k'$ of the complex tetrahedral surface $\mathcal{T}_0$ are mainly given by intersecting del Pezzo surfaces dP_k. This property is expected from general arguments since the blowing of the tetrahedron

$$\mathcal{T} = \cup_a \mathcal{D}_a \tag{5.5}$$

together with the relations (5.2-5.4), amounts to blowing the divisors $\mathcal{D}_a$. But these divisors homeomorphic to CP^2s embedded in CP^3. We have checked this property for the toric blow ups type; but we don't have yet the answer whether this result is true as well for the non toric blow ups.
b) Toric geometry has a nice feature which can be used in the engineering of F-theory GUT- like models building. In going from the faces to the vertices of the tetrahedron, cycles of the toric fibers shrink down as shown in the following table

$$
\begin{array}{llccccc}
\text{tetrahedron } \Delta_{\mathcal{T}_n} & : & \text{faces} & & \text{edges} & & \text{vertices} \\
\text{toric fibers} & : & T^2 & \rightarrow & S^1 & \rightarrow & 0 \\
\text{toric symmetries} & : & U(1) \times U(1) & \rightarrow & U(1) & \rightarrow & 0
\end{array}
\tag{5.6}
$$

In the field theory language, these shrinking generate massless modes which may be interpreted in terms of massless gauge fields and so gauge symmetry enhancements at the level of the 4D space time effective field theory. More precisely, given a gauge symmetry G_r that is visible 4D space time, the gauge symmetry associated with the faces $\mathcal{D}_a$ of the tetrahedron and its blow ups would be

$$
G_r \times U(1) \times U(1), \tag{5.7}
$$

where the $U(1)$ factors may be interpreted in terms of branes wrapping cycles in the toric fibration. The bulk invariance (5.7) gets enhanced to a $G_{r+1} \times U(1)$ invariance on the edges $\Sigma_{(ab)}$ and further to a G_{r+2} gauge symmetry at the vertices $P_{(abc)}$.
In the case where $G_r = SU(5)$ for example, the gauge enhanced symmetry on the edges could be either $SU(6)$ or $SO(10)$ and at the vertices it may be one of the following enhanced gauge symmetries

$$
SU(7) \quad , \quad SO(12) \quad , \quad E_6 \quad . \tag{5.8}
$$

We end this conclusion by adding a comment regarding the way the tetrahedron surface and its blown up cousins $\mathcal{T}_n$ and $\mathcal{T}'_k$ could be used in practice. They should be thought of as the base surface of the elliptically K3 fibered Calabi-Yau four- folds in the F-theory compactification to 4D space time,

$$
\begin{array}{ccc}
\text{Y} & \rightarrow & \text{CY4} \\
& & \downarrow \pi_n \\
& & \mathcal{T}_n
\end{array}
\tag{5.9}
$$

These complex surfaces are wrapped by seven branes with intersections along the edges and at the vertices. On the edges localize chiral matters $\Phi^a_{R_a}$ in bi-fundamental representations while at the vertices of the toric graphs live a 4D $\mathcal{N} = 1$ supersymmetric Yukawa couplings with chiral potential,

$$
W_{Yuk} = \int d^4x d^2\theta \left(\sum_{a<b<c} \frac{\lambda_{abc}}{3} \Phi^a_{R_a} \Phi^b_{R_b} \Phi^c_{R_c} \right). \tag{5.10}
$$

where the complex numbers λ_{abc} are Yukawa coupling constants. In this 4D $\mathcal{N} = 1$ superspace relation, $\Phi^a_{R_a}$ stands for a family of chiral superfields transforming in representations R_a of the gauge group $G_r \times U(1) \times U(1)$ with the constraint equation

$$
R_a \otimes R_b \otimes R_c = \mathbf{1} \oplus \left(\bigoplus_i \mathrm{f}^i_{abc} R_i \right), \tag{5.11}
$$

where f^i_{abc} are some positive integers capturing the multiplicity of the representation R_i. If relaxing the BHV model to include as well those unrealistic F-theory *GUT-like* models by allowing exotic fields, the blow ups geometries $\mathcal{T}_n$ and $\mathcal{T}'_k$ may be used to engineer effective quiver gauge theories that are embedded in F-theory on Calabi-Yau 4-folds. In this view, by using for instance the blown surface of figure (8) and taking

$$G_r = SU(5)\,, \tag{5.12}$$

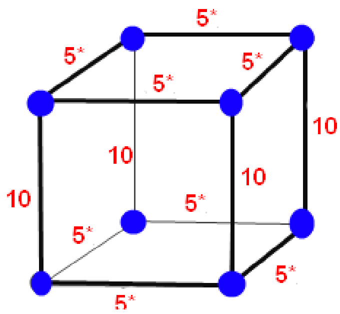
we may engineer various 4D $\mathcal{N} = 1$ supersymmetric $SU(5)$ quiver gauge models like the two ones depicted on the figures (9). For these examples, chiral matters $\Phi^a_{R_a}$ localizing on each on of the twelve edges $\Sigma_{(ab)}$ transform in on of the following SU(5) representations

$$R_a \quad \equiv \quad 1 \quad , \quad 5 \quad , \quad 5^* \quad , \quad 10 \quad . \tag{5.13}$$

These representations R_a, which carry also charges (q_a, q'_a) under the $U(1) \times U(1)$ toric symmetry, describe the usual chiral matter 5^*_M and 10_M as well as its Higgs fields 5_{up} and 5^*_{down} of the $SU(5)$ GUT model but also exotic matter.
Yukawa couplings localizing at the eight vertices $P_{(abc)}$ of the graph (8) are given by

$$5^*_a \times 5^*_b \times 10_c \quad , \quad 5^*_a \times 5_b \times 1_c \quad , \tag{5.14}$$

where the singlets 1_c stand for right neutrinos-like and right leptons- like. The geometric engineering of such kind of quiver gauge theories will be extensively developed in [20].



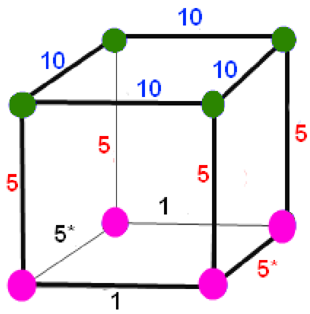


Figure 9: On left we have the quiver gauge diagram of an SU(5) GUT-like model with eight Yukawa couplings type $5^*_a \times 5^*_b \times 10_c$. On the right the quiver graph of an SU(5) GUT-like model with four Yukawa couplings type $5^*_a \times 5^*_b \times 10_c$ and four others of type $5^*_a \times 5_b \times 1_c$. These two $SU(5)$ GUT-like models have exotic fields.

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